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Rothberger's property in all finite powers

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ABSTRACT

A space X has the Rothberger property in all finite powers if, and only if, its collection of ω-covers has Ramseyan properties.

For $s \in [\mathbb{N}]^{<\aleph_0}$ and for $B \in [\mathbb{N}]^{\aleph_0}$ use $s < B$ to denote that $s = \emptyset$ or $\max(s) < \min(B)$. For $s < B$ define $[s, B] = \{s \cup C \in [\mathbb{N}]^{\aleph_0}: s < C \subseteq B\}$. The family $\{[s, B]: s \subset \mathbb{N}$ finite and $s < B \in [\mathbb{N}]^{\aleph_0}\}$ forms a basis for a topology on $[\mathbb{N}]^{\aleph_0}$. This is the *Ellentuck topology* on $[\mathbb{N}]^{\aleph_0}$ and was introduced in [3].

Recall that a subset N of a topological space is *nowhere dense* if there is for each nonempty open set U of the space a nonempty open subset $V \subset U$ such that $N \cap V = \emptyset$. And N is said to be *meager* if it is a union of countably many nowhere dense sets. A subset of a topological space is said to have the *Baire property* if it is of the form $(U \setminus M) \cup (M \setminus U)$ for some open set U and some meager set M.

Theorem 1 *(Ellentuck). For a set $R \subset [\mathbb{N}]^{\aleph_0}$ the following are equivalent*:

(1) *R has the Baire property in the Ellentuck topology.*
(2) *For each finite set $s \subset \mathbb{N}$ and for each infinite set $S \subset \mathbb{N}$ with $s < S$ there is an infinite set $T \subset S$ such that either $[s, T] \subset R$, or else $[s, T] \cap R = \emptyset$.*

The proof of (1) ⇒ (2) is nontrivial but uses only the techniques of Galvin and Prikry [5]. Galvin and Prikry proved a precursor of Theorem 1: If R is a Borel set in the topology inherited from $2^{\mathbb{N}}$ via representing sets by their characteristic functions, then R has property (2) in Theorem 1. Silver and Mathias subsequently gave metamathematical proofs that analytic sets (in the $2^{\mathbb{N}}$-topology) have this property. Theorem 1 at once yields all these prior results. The original papers [3] and [5] give a nice overview of these facts, and more.

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When a subset $\mathcal{X}$ of $[\mathbb{N}]^{\aleph_0}$ inherits the Ellentuck topology from $[\mathbb{N}]^{\aleph_0}$, we shall speak of "$\mathcal{X}$ with the Ellentuck topology". For A an abstract countably infinite set define the Ellentuck topology on $[A]^{\aleph_0}$ by fixing a bijective enumeration $(a_n\colon n \in \mathbb{N})$ of A and by defining for s and T nonempty subsets of A:

$$s < T \quad \text{if: } a_n \in s \text{ and } a_m \in T \quad \Rightarrow \quad n < m.$$

With the relation $s < T$ defined, define the Ellentuck topology on $[A]^{\aleph_0}$ as above. For $B \subseteq A$ and for finite set $s \subseteq A$ we write $B|s$ for $\{a_n \in B\colon s < \{a_n\}\}$.

For families $\mathcal{A}$ and $\mathcal{B}$ we now define a sequence of statements:

$\mathsf{E}(\mathcal{A}, \mathcal{B})$: For each countably infinite $A \in \mathcal{A}$ and for each set $R \subset [A]^{\aleph_0} \cap \mathcal{B}$ the implication (1) $\Rightarrow$ (2) holds, where:
(1) R has the Baire property in the Ellentuck topology on $[A]^{\aleph_0} \cap \mathcal{B}$.
(2) For each $S \subset A$ with $S \in \mathcal{A}$ and each finite subset s of A, there is an infinite $B \subset S|s$ with $B \in \mathcal{B}$ such that $[s, B] \cap \mathcal{B} \subseteq R$ or $[s, B] \cap \mathcal{B} \cap R = \emptyset$.

Thus, $\mathsf{E}([\mathbb{N}]^{\aleph_0}, [\mathbb{N}]^{\aleph_0})$ is Ellentuck's theorem.

$\mathsf{GP}(\mathcal{A}, \mathcal{B})$: For each countably infinite $A \in \mathcal{A}$ and each $R \subset [A]^{\aleph_0} \cap \mathcal{B}$ the implication (1) $\Rightarrow$ (2) holds:
(1) R is open in the $2^{\mathbb{N}}$ topology on $[A]^{\aleph_0} \cap \mathcal{B}$.
(2) For each $S \in [A]^{\aleph_0} \cap \mathcal{A}$ there is a set $B \in [S]^{\aleph_0} \cap \mathcal{B}$ such that either $([B]^{\aleph_0} \cap \mathcal{B}) \subseteq R$, or else $[B]^{\aleph_0} \cap \mathcal{B} \cap R = \emptyset$.

Thus, $\mathsf{GP}([\mathbb{N}]^{\aleph_0}, [\mathbb{N}]^{\aleph_0})$ is part of the Galvin–Prikry theorem.

Definition 1. A subset $\mathcal{S}$ of $[A]^{<\aleph_0}$ is:

(1) dense if for each $B \in [A]^{\aleph_0} \cap \mathcal{A}$, $\mathcal{S} \cap [B]^{<\aleph_0} \neq \emptyset$.
(1) thin if no element of $\mathcal{S}$ is an initial segment of another element of $\mathcal{S}$.

The following is an abstract formulation of Galvin's generalization of Ramsey's Theorem, announced in [4] and in [5] derived from Theorem 1 there:

$\mathsf{FG}(\mathcal{A}, \mathcal{B})$: For each countably infinite $A \in \mathcal{A}$ and for each dense set $\mathcal{S} \subset [A]^{<\aleph_0}$ there is a $B \in [A]^{\aleph_0} \cap \mathcal{B}$ such that each $C \in [B]^{\aleph_0} \cap \mathcal{B}$ has an initial segment in $\mathcal{S}$.

In this notation Galvin's generalization of Ramsey's theorem reads that $\mathsf{FG}([\mathbb{N}]^{\aleph_0}, [\mathbb{N}]^{\aleph_0})$. Similarly, the following is an abstract formulation of Nash–Williams' theorem:

$\mathsf{NW}(\mathcal{A}, \mathcal{B})$: For each countably infinite $A \in \mathcal{A}$ and for each thin family $\mathcal{T} \subset [A]^{<\aleph_0}$ and for each n, and each partition $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 \cup \cdots \cup \mathcal{T}_n$ there is a $B \in [A]^{\aleph_0} \cap \mathcal{B}$ and an $i \in \{1, \ldots, n\}$ such that $[B]^{<\aleph_0} \cap \mathcal{T} \subseteq \mathcal{T}_i$.

In this notation Nash–Williams' theorem reads that $\mathsf{NW}([\mathbb{N}]^{\aleph_0}, [\mathbb{N}]^{\aleph_0})$.

$\mathcal{A} \to (\mathcal{B})^n_k$: For positive integers n and k and for each countable $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{1, \ldots, k\}$ there is a $B \in [A]^{\aleph_0} \cap \mathcal{B}$ and an $i \in \{1, \ldots, k\}$ such that f has value i on $[B]^n$.

In this notation Ramsey's theorem reads: For each n and k, $[\mathbb{N}]^{\aleph_0} \to ([\mathbb{N}]^{\aleph_0})^n_k$.

An open cover $\mathcal{U}$ of a topological space X is said to be an ω-cover if $X \notin \mathcal{U}$, but there is for each finite set $F \subset X$ a $U \in \mathcal{U}$ with $F \subseteq U$. The symbol Ω_X denotes the collection of ω-covers of X. The symbol $\mathcal{O}_X$ denotes the collection of open covers of X. In [9] Rothberger introduced the following covering property: For each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of open covers of X there is a sequence $(U_n\colon n \in \mathbb{N})$ such that each $U_n \in \mathcal{U}_n$, and $\{U_n\colon n \in \mathbb{N}\}$ is a cover of X. The symbol $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$ denotes this statement. The corresponding statement for ω-covers of X, $\mathsf{S}_1(\Omega_X, \Omega_X)$, was introduced in [10] by Sakai. It states: For each sequence $(\mathcal{U}_n\colon n \in \mathbb{N})$ of ω-covers of X there is a sequence $(U_n\colon n \in \mathbb{N})$ such that each $U_n \in \mathcal{U}_n$, and $\{U_n\colon n \in \mathbb{N}\}$ is an ω cover for X. Sakai proved that X has $\mathsf{S}_1(\Omega_X, \Omega_X)$ if, and only if, all finite powers of X have $\mathsf{S}_1(\mathcal{O}_X, \mathcal{O}_X)$. According to Gerlits and Nagy [6] a space is said to be an *ϵ-space* if each ω-cover contains a countable subset which still is an ω-cover. A space is an ϵ-space if and only if it has the Lindelöf property in all finite powers—see [6] for details. In this paper we prove:

Theorem 2. *For an ϵ-space X, the following are equivalent*:

(1) $\mathsf{S}_1(\Omega_X, \Omega_X)$.
(2) $\mathsf{E}(\Omega_X, \Omega_X)$.

(3) $\mathsf{GP}(\Omega_X, \Omega_X)$.
(4) $\mathsf{FG}(\Omega_X, \Omega_X)$.
(5) $\mathsf{NW}(\Omega_X, \Omega_X)$.
(6) *For all n and k,* $\Omega_X \to (\Omega_X)^n_k$.
(7) $\Omega_X \to (\Omega_X)^2_2$.

1. The proof of $\mathsf{S}_1(\Omega_X, \Omega_X) \Rightarrow \mathsf{E}(\Omega_X, \Omega_X)$

Assume that X has property $\mathsf{S}_1(\Omega_X, \Omega_X)$. Fix a countable $A \in \Omega_X$ and fix a set $R \subset [A]^{\aleph_0} \cap \Omega_X$. For the remainder of the argument, fix a bijective enumeration of A, say $(a_n\colon n \in \mathbb{N})$. Sets of the form $[s, C] = \{D\colon s < C$ and $s \subset D \subseteq s \cup C\}$ constitute a basis for the Ellentuck topology on $[A]^{\aleph_0}$.

Definition 2. For a finite set $s \subset A$ and for $B \in [A|s]^{\aleph_0} \cap \Omega_X$:

(1) B accepts s if $[s, B] \cap \Omega_X \subseteq R$.
(2) B rejects s if no $C \in [B]^{\aleph_0} \cap \Omega_X$ accepts s.

Lemma 3 will be used without special reference:

Lemma 3. *Let a finite set $s \subset A$ and a set $B \in [A|s]^{\aleph_0} \cap \Omega_X$ be given*:

(1) *B accepts s if, and only if, each $C \in [B]^{\aleph_0} \cap \Omega_X$ accepts s.*
(2) *B rejects s if, and only if, each $C \in [B]^{\aleph_0} \cap \Omega_X$ rejects s.*

Lemma 4. *For each finite set $s \subset A$, there is a $B \in [A|s]^{\aleph_0} \cap \Omega_X$ such that B accepts s or B rejects s.*

Proof. If $A|s$ does not reject s, choose a $B \in [A|s]^{\aleph_0} \cap \Omega_X$ accepting s. □

Lemma 5. *Let $t \subset A$ be a finite set. Let $B \in [A]^{\aleph_0} \cap \Omega_X$ be such that for each finite set $s \subset (t \cup B)$, $B|s$ accepts s or $B|s$ rejects s. If $B|t$ rejects t then $C = \{u \in B\colon B|(t \cup \{u\})$ rejects $t \cup \{u\}\}$ is a member of Ω_X.*

Proof. Suppose not. Then $D = t \cup (B \setminus C) \in \Omega_X$, and for each $u \in D|t$, $B|(t \cup \{u\})$ accepts $t \cup \{u\}$. Thus for each $u \in D|t$, $D|(t \cup \{u\})$ accepts $t \cup \{u\}$. This means that $[t, D|t] = \bigcup_{u \in D}[t \cup \{u\}, D|(t \cup \{u\})] \subseteq R$, and so $D|t$ accepts t. This contradicts Lemma 3(2) since $D \in [B]^{\aleph_0} \cap \Omega_X$ and $B|t$ rejects t. □

1.1. ω-covers accepting or rejecting all finite subsets

The game $\mathsf{G}_1(\Omega_X, \Omega_X)$ is played as follows: Players ONE and TWO play an inning per positive integer. In the nth inning ONE first chooses an $O_n \in \Omega_X$; TWO responds with a $T_n \in O_n$. A play $O_1, T_1, \ldots, O_n, T_n, \ldots$ is won by TWO if $\{T_n\colon n \in \mathbb{N}\} \in \Omega_X$; else, ONE wins. It was shown in [13] that

Theorem 6. *For a topological space Y the following are equivalent*:

(1) *Y has property $\mathsf{S}_1(\Omega_Y, \Omega_Y)$.*
(2) *ONE has no winning strategy in $\mathsf{G}_1(\Omega_Y, \Omega_Y)$.*

Theorem 7. *If Y has property $\mathsf{S}_1(\Omega_Y, \Omega_Y)$, then for each finite set $t \subset A$ and for each $B \in [A|t]^{\aleph_0} \cap \Omega_Y$ there is a $C \in [B]^{\aleph_0} \cap \Omega_Y$ such that for each finite set $s \subset t \cup C$, $C|s$ accepts s or $C|s$ rejects s.*

Proof. Let t and $B \in [A|t]^{\aleph_0} \cap \Omega_Y$ be given. Define a strategy σ for ONE of $\mathsf{G}_1(\Omega_Y, \Omega_Y)$ as follows:

Enumerate the set of all subsets of t as $\{t_1, \ldots, t_n\}$. Using Lemma 4 recursively choose $B_1 \supset B_2 \supset \cdots \supset B_n$ in $[B]^{\aleph_0} \cap \Omega_Y$ such that for each i, B_i accepts t_i or B_i rejects t_i. Then define:

$$\sigma(\emptyset) = B_n.$$

If TWO now chooses $T_1 \in \sigma(\emptyset)$ then use Lemma 4 in the same way to choose

$$\sigma(T_1) \in \left[\sigma(\emptyset)|\{T_1\}\right]^{\aleph_0} \cap \Omega_Y$$

such that for each set $F \subset t \cup \{T_1\}$, $\sigma(T_1)$ accepts F, or rejects F.

When TWO responds with $T_2 \in F(T_1)$, enumerate the subsets of $t \cup \{T_1, T_2\}$ as $(t_1, \dots, t_n)$ say, and choose by Lemma 4 sets $B_1, \dots B_n \in [\sigma(T_1)|\{T_2\}]^{\aleph_0} \cap \Omega_X$ such that B_j accepts t_j or B_j rejects t_j for $1 \leqslant j \leqslant n$ and $B_j \subset B_{j-1}$. Finally put

$$\sigma(T_1, T_2) = B_n.$$

Note that for each finite subset F of $t \cup \{T_1, T_2\}$, $\sigma(T_1, T_2)$ accepts F or rejects it.

It is clear how player ONE's strategy is defined. By Theorem 6 σ is not a winning strategy for ONE. Consider a σ-play lost by ONE, say

$$\sigma(\emptyset), T_1, \sigma(T_1), T_2, \sigma(T_1, T_2), \dots, T_n, \sigma(T_1, \dots, T_n), \dots$$

Then $C = t \cup \{T_n\colon n \in \mathbb{N}\} \subset B$ is an element of Ω_Y.

We claim that for each finite subset s of $t \cup C$, $C|s$ accepts s or $C|s$ rejects s. For consider such a s. If $s \subseteq t$, then as $C \subset F(\emptyset)$ and $F(\emptyset)$ accepts or rejects s, also C does. If $s \nsubseteq t$, then put $n = \max\{m\colon T_m \in s\}$. Then s is a subset of $t \cup \{T_1, \dots, T_n\}$, so that s is accepted or rejected by $\sigma(T_1, \dots, T_n)$. But $C|s \subseteq \sigma(T_1, \dots, T_n)$, and so $C|s$ accepts or rejects s. □

1.2. Completely Ramsey sets

The subset R of $[A]^{\aleph_0} \cap \Omega_X$ is said to be *completely Ramsey* if there is for each finite set $s \subset A$ and for each $B \in [A|s]^{\aleph_0} \cap \Omega_X$ a set $C \in [B]^{\aleph_0} \cap \Omega_X$ such that

(1) either $([s, C] \cap \Omega_X) \subseteq R$,
(2) or else $([s, C] \cap \Omega_X) \cap R = \emptyset$.

Lemma 8. *If R and S are completely Ramsey subsets of $[A]^{\aleph_0} \cap \Omega_X$, then so is $R \cup S$.*

Proof. Let a finite set $s \subset A$ and $B \in [A|s]^{\aleph_0} \cap \Omega_X$ be given. Since R is completely Ramsey, choose $C \in [B]^{\aleph_0} \cap \Omega_X$ such that $([s, C] \cap \Omega_X) \subset R$, or $([s, C] \cap \Omega_X) \cap R = \emptyset$. If the former hold we are done. In the latter case, since S is completely Ramsey, choose $D \in [C]^{\aleph_0} \cap \Omega_X$ such that $([s, D] \cap \Omega_X) \subseteq S$, or $([s, D] \cap \Omega_X) \cap S = \emptyset$. In either case the proof is complete. □

The following lemma is obviously true.

Lemma 9. *If R is completely Ramsey, then so is $([A]^{\aleph_0} \cap \Omega_X) \setminus R$.*

Corollary 10. *If R and S are completely Ramsey subsets of $[A]^{\aleph_0} \cap \Omega_X$, then so is $R \cap S$.*

Proof. Lemmas 8 and 9, and De Morgan's laws. □

1.3. Open sets in the Ellentuck topology

We are still subject to the hypothesis that X satisfies $\mathsf{S}_1(\Omega_X, \Omega_X)$.

Lemma 11. *For each finite set $t \subset A$ and for each $B \in [A|t]^{\aleph_0} \cap \Omega_X$ such that for each finite subset F of $t \cup B$, $B|F$ accepts, or rejects F the following holds: For each finite set $s \subset t \cup B$ such that $B|s$ rejects s, there is a $C \in [B|s]^{\aleph_0} \cap \Omega_X$ such that for each finite set $F \subset C$, $C|F$ rejects $s \cup F$.*

Proof. Fix B and s as in the hypotheses. Define a strategy σ for ONE in $\mathsf{G}_1(\Omega_X, \Omega_X)$ as follows: By Lemma 5

$$\sigma(\emptyset) = \bigl\{U \in B\colon s < \{U\} \text{ and } B|\{U\} \text{ rejects } s \cup \{U\}\bigr\} \in \Omega_X.$$

Notice that $\sigma(\emptyset)$ accepts or rejects each of its finite subsets, it rejects s, and for each $U \in \sigma(\emptyset)$, $\sigma(\emptyset)|\{U\}$ rejects $s \cup \{U\}$.

If TWO now chooses $T_1 \in \sigma(\emptyset)$, then by Lemma 5

$$\sigma(T_1) = \bigl\{U \in \sigma(\emptyset) \setminus \{T_1\}\colon \sigma(\emptyset)|F \text{ rejects } s \cup F \text{ for each finite } F \subset \{T_1, U\}\bigr\}$$

is in Ω_X. As before, $\sigma(T_1)$ accepts or rejects each of its finite subsets, and for any $U \in \sigma(T_1)$, for each finite subset F of $\{U, T_1\}$, $\sigma(T_1)|F$ rejects $s \cup F$.

If next TWO chooses $T_2 \in \sigma(T_1)$, then by Lemma 5

$$\sigma(T_1, T_2) = \bigl\{U \in \sigma(T_1) \setminus \{T_2\}\colon \sigma(T_1)|F \text{ rejects } s \cup F \text{ for any finite } F \subset \{T_1, T_2, U\}\bigr\}$$

is an element of Ω_X.

Continuing in this way we define a strategy σ for ONE in $\mathsf{G}_1(\Omega_X, \Omega_X)$. Since X satisfies $\mathsf{S}_1(\Omega_X, \Omega_X)$, σ is not a winning strategy for ONE. Consider a σ-play lost by ONE, say:

$$\sigma(\emptyset),\ T_1,\ \sigma(T_1),\ T_2,\ \sigma(T_1, T_2),\ T_3,\ \sigma(T_1, T_2, T_3),\ \ldots.$$

Put $C = \{T_n\colon n \in \mathbb{N}\}$. Then $C \in [B|s]^{\aleph_0} \cap \Omega_X$. We claim that for each finite set $F \subset C$, $C|F$ rejects $s \cup F$.

For choose a finite set $F \subset C$. Then $F \cap s = \emptyset$. Fix $n = \max\{m\colon T_m \in F\}$. Then $C|F \subset \sigma(T_1, \ldots, T_n)$, and the latter rejects $s \cup F$ for all finite subsets F of $\{T_1, \ldots, T_n\}$. Thus $C|F$ rejects $s \cup F$. □

Theorem 12. *If X has property $\mathsf{S}_1(\Omega_X, \Omega_X)$, then every open subset of $[A]^{\aleph_0} \cap \Omega_X$ is completely Ramsey.*

Proof. Let $R \subset [A]^{\aleph_0} \cap \Omega_X$ be open in this subspace. Consider a finite set $s \subset A$ and a $B \in [A|s]^{\aleph_0} \cap \Omega_X$. Since $(X, d) \models \mathsf{S}_1(\Omega_X, \Omega_X)$, choose by Theorem 7 a $C \in [B]^{\aleph_0} \cap \Omega_X$ such that for each finite set $F \subset (s \cup C)$, $C|F$ accepts or rejects F.

If C accepts s then we have $[s, C] \cap \Omega_X \subseteq R$, and we are done. Thus, assume that C does not accept s. Then C rejects s, and we choose by Lemma 11 a $D \in [C|s]^{\aleph_0} \cap \Omega_X$ such that for each finite subset F of D, $D|F$ rejects $s \cup F$.

We claim that $([s, D] \cap \Omega_X) \cap R = \emptyset$. For suppose not. Choose $E \in [s, D] \cap \Omega_X \cap R$. Since R is open, choose an Ellentuck neighborhood of E contained in R, say $[t, K] \cap \Omega_X$. Then we have $s \subset E \subset s \cup D$ and $t \subset E \subset t \cup K$. But then $s \cup t \subset E \subset t \cup K$ and $[s \cup t, K|s] \subset R$, whence also $[s \cup t, E|(s \cup t)] \subset R$. But then $E|(s \cup t)$ accepts $s \cup t$ where t is a finite subset of $s \cup D$, and $E|(s \cup t) \subset D|t$, and $D|t$ rejects $s \cup t$, a contradiction. □

1.4. Meager subsets in the Ellentuck topology

If the subset R of $[A]^{\aleph_0} \cap \Omega_X$ is nowhere dense in the topology, then for each $B \in [A]^{\aleph_0} \cap \Omega_X$ and for each finite set $s \subset A$, $B|s$ rejects s. We now examine the meager subsets of $[A]^{\aleph_0} \cap \Omega_X$.

Lemma 13. *If R is nowhere dense, then there is for each $B \in [A]^{\aleph_0} \cap \Omega_X$ and each finite set $t \subset A$ a set $C \in [B|t]^{\aleph_0} \cap \Omega_X$ such that for each finite set $s \subset t \cup C$, $C|s$ rejects s.*

Proof. Since R is nowhere dense, no ω-cover contained in A can accept a finite set. Thus each ω-cover contained in A rejects each finite subset of A. □

Lemma 14. *Assume $\mathsf{S}_1(\Omega_X, \Omega_X)$. If R is a closed nowhere dense subset of $[A]^{\aleph_0} \cap \Omega_X$ then there is for each finite subset $s \subset A$ and for each $B \in [A|s]^{\aleph_0} \cap \Omega_X$ a $C \in [B]^{\aleph_0} \cap \Omega_X$ such that $[s, C] \cap R = \emptyset$.*

Proof. First, note that closed nowhere dense subsets are complements of open dense sets. By Theorem 12, each open set is completely Ramsey. By Lemma 9 each closed, nowhere dense set is completely Ramsey. By Lemma 13 the rest of the statement follows. □

By taking closures, the preceding lemma implies:

Corollary 15. *Assume $\mathsf{S}_1(\Omega_X, \Omega_X)$. If R is a nowhere dense subset of $[A]^{\aleph_0} \cap \Omega_X$ then there is for each finite subset $s \subset A$ and for each $B \in [A|s]^{\aleph_0} \cap \Omega_X$ a $C \in [B]^{\aleph_0} \cap \Omega_X$ such that $[s, C] \cap R = \emptyset$.*

And now we prove:

Theorem 16. *Assume $\mathsf{S}_1(\Omega_X, \Omega_X)$. For a subset N of $[A]^{\aleph_0} \cap \Omega_X$ the following are equivalent*:

(1) *N is nowhere dense.*
(2) *N is meager.*

Proof. We must show that (2) ⇒ (1). Thus, assume that N is meager and write $N = \bigcup_{n \in \mathbb{N}} N_n$, where for each n we have $N_n \subseteq N_{n+1}$, and N_n is nowhere dense in $[A]^{\aleph_0} \cap \Omega_X$. Consider any basic open set $[s, B]$ of $[A]^{\aleph_0} \cap \Omega_X$. Define a strategy σ for ONE in the game $\mathsf{G}_1(\Omega_X, \Omega_X)$ as follows:

Since N_1 is nowhere dense, choose by Corollary 15 an $O_1 \in [B]^{\aleph_0} \cap \Omega_X$ with $[s, O_1] \cap N_1 = \emptyset$. Define $\sigma(\emptyset) = O_1$.

When TWO chooses $T_1 \in \sigma(\emptyset)$ choose by Corollary 15 an $O_2 \in [\sigma(\emptyset)|\{T_1\}]^{\aleph_0} \cap \Omega_X$ with $[s, O_2] \cap N_2 = \emptyset$, and define $\sigma(T_1) = O_2$.

Now when TWO chooses $T_2 \in \sigma(T_1)$, find by Corollary 15 an $O_3 \in [\sigma(T_1)|\{T_2\}]^{\aleph_0} \cap \Omega_X$ with $[s, O_3] \cap N_3 = \emptyset$, and define $\sigma(T_1, T_2) = O_3$.

It is clear how to define ONE's strategy σ. By Theorem 6 F is not a winning strategy for ONE. Consider a play

$$\sigma(\emptyset),\ T_1,\ \sigma(T_1),\ T_2,\ \ldots,\ \sigma(T_1,\ldots,T_n),\ T_{n+1},\ \ldots$$

lost by ONE. Put $C=\{T_n:\ n\in\mathbb{N}\}$. Then $C\in[B]^{\aleph_0}\cap\Omega_X$. Observe that by the definition of σ we have for each k and each finite set $F\subset\{T_1,\ldots,T_k\}$ that $[s\cup F,\sigma(T_1,\ldots,T_k)]\cap N_k=\emptyset$.

Claim. $[s,C]\cap N=\emptyset$.

For suppose that instead $[s,C]\cap N\neq\emptyset$. Choose $V\in[s,C]\cap N$, and then choose m so that $V\in N_m$. Choose the least $k>m$ with $T_k\in V|s$. This is possible because s is finite. Observe also that $s\subseteq V\subseteq s\cup C=s\cup\{T_j:\ j\in\mathbb{N}\}$. Put $F=V\cap\{T_1,\ldots,T_k\}$. Thus we have that $[s\cup F,V|F]\cap N_k\neq\emptyset$, which contradicts the fact that $V|F\subset\sigma(T_1,\ldots,T_k)$, and $[s\cup F,\sigma(T_1,\ldots,T_k)]\cap N_k=\emptyset$. This completes the proof of the claim. □

Using Lemmas 8 and 9 and Corollary 10 we have:

Theorem 17. *Suppose X satisfies $\mathsf{S}_1(\Omega_X,\Omega_X)$. Then for each $A\in\Omega_X$, every subset of $[A]^{\aleph_0}\cap\Omega_X$ which has the Baire property is completely Ramsey.*

2. The proof of $\mathsf{E}(\Omega_X,\Omega_X)\Rightarrow\mathsf{S}_1(\Omega_X,\Omega_X)$

Note that a set open in the $2^{\mathbb{N}}$ topology is also open in the Ellentuck topology. The implication (2) ⇒ (3) of Theorem 2 follows from this remark. Now we start with (3).

Lemma 18. *Assume $\mathsf{GP}(\Omega_X,\Omega_X)$. Then $\mathsf{FG}(\Omega_X,\Omega_X)$ holds.*

Proof. Let $\mathcal{S}\subset[A]^{<\aleph_0}$ be dense and define $\mathcal{I}$ to be the set $\{D\in[A]^{\aleph_0}\cap\Omega_X:\ D$ has an initial segment in $\mathcal{S}\}$. Then we have:

$$\mathcal{I}=\bigcup\big\{[s,D|s]:\ s\in\mathcal{S},\ D\in[A]^{\aleph_0}\cap\Omega_X\text{ and s an initial segment of }D\big\}$$

is a $2^{\mathbb{N}}$-open subset of $[A]^{\aleph_0}\cap\Omega_X$. Choose a $B\in[A]^{\aleph_0}\cap\Omega_X$ such that $[B]^{\aleph_0}\cap\Omega_X\subset\mathcal{I}$, or $[B]^{\aleph_0}\cap\Omega_X\cap\mathcal{I}=\emptyset$. But the second alternative implies the contradiction that $[B]^{<\aleph_0}\cap\mathcal{S}=\emptyset$. It follows that the first alternative holds. □

Theorem 19. *Assume $\mathsf{FG}(\Omega_X,\Omega_X)$. Then $\mathsf{NW}(\Omega_X,\Omega_X)$ holds.*

Proof. Fix a thin family $\mathcal{T}\subset[A]^{<\aleph_0}$ and positive integer n, and a partition $\mathcal{T}=\mathcal{T}_1\cup\mathcal{T}_2\cup\cdots\cup\mathcal{T}_n$. We may assume $n=2$. If $\mathcal{T}_1$ is not dense, we can choose $B\in[A]^{\aleph_0}\cap\Omega_X$ such that $[B]^{<\aleph_0}\cap\mathcal{T}\subseteq\mathcal{T}_2$. Thus, assume $\mathcal{T}_1$ is dense. Choose, by the hypothesis, a $B\in[A]^{\aleph_0}\cap\Omega_X$ such that for each $C\in[B]^{\aleph_0}\cap\Omega_X$, some initial segment of C is in $\mathcal{T}_1$.

Consider any $s\in\mathcal{T}\cap[B]^{<\aleph_0}$, and put $D=s\cup(B|s)$. Then s is an initial segment of D, and $D\in[B]^{\aleph_0}\cap\Omega_X$, and so some initial segment of D, say t, is in $\mathcal{T}_1$. Since both t and s are initial segments of D and are both in $\mathcal{T}$, and since $\mathcal{T}$ is thin, we have $s=t$, and so $s\in\mathcal{T}_1$. Consequently we have $[B]^{<\aleph_0}\cap\mathcal{T}\subseteq\mathcal{T}_1$. □

Theorem 20. *Assume that $\mathsf{NW}(\Omega_X,\Omega_X)$ holds. Then for each n and k we have $\Omega_X\to(\Omega_X)^n_k$.*

Proof. Let $A\in\Omega_X$ be countable. Let positive integers n and k be given. Put $\mathcal{T}=[A]^n$. Then $\mathcal{T}$ is thin. Apply the hypothesis. □

The following theorem was proven in [7] (Theorem 6.1) and [12] (Theorem 24).[1] It, together with the above sequence of implications, completes the proof of Theorem 2.

Theorem 21. *The following are equivalent*:

(1) *For each n and k, $\Omega_X\to(\Omega_X)^n_k$*
(2) $X\models\mathsf{S}_1(\Omega_X,\Omega_X)$.

[1] See Appendix A.

3. Remarks

The results above are given for Ω, but a study of the proofs will reveal that these equivalences hold for several other families $\mathcal{A}$. The main requirements on $\mathcal{A}$ are that each element of $\mathcal{A}$ has a countable subset in $\mathcal{A}$, that for each k $\mathcal{A} \to (\mathcal{A})^1_k$ holds, and that $\mathsf{S}_1(\mathcal{A}, \mathcal{A})$ is equivalent to ONE not having a winning strategy in $\mathsf{G}_1(\mathcal{A}, \mathcal{A})$, and that this is equivalent to $\mathcal{A} \to (\mathcal{A})^2_2$. Though this general treatment can be given without much additional effort, I preferred to illustrate the equivalences using a well-known concrete example, because of the connections of this example with forcing (pointed out below) and with the famous Borel Conjecture. Here are a few examples of such families $\mathcal{A}$:

For a topological space X and an element $x \in X$, define $\Omega_x = \{A \subset X \setminus \{x\}\colon\ x \in \overline{A}\}$. According to [10] X has strong countable fan tightness at x if the selection principle $\mathsf{S}_1(\Omega_x, \Omega_x)$ holds. Consider for a Tychonoff space X the subspace of the Tychonoff product $\Pi_{x\in X}\mathbb{R}$ consisting of the continuous functions from X to $\mathbb{R}$. The symbol $\mathsf{C}_p(X)$ denotes this subspace with the inherited topology. Since $\mathsf{C}_p(X)$ is homogeneous, the truth of $\mathsf{S}_1(\Omega_f, \Omega_f)$ at some point f implies the truth of $\mathsf{S}_1(\Omega_f, \Omega_f)$ at any point f. Thus we may confine attention to $\Omega_{\mathbf{o}}$, where $\mathbf{o}$ is the function which is zero on X. Using the techniques above one can prove:

Theorem 22. *For a Tychonoff space X the following are equivalent for* $\mathsf{C}_p(X)$:

(1) $\mathsf{S}_1(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(2) $\mathsf{E}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(3) $\mathsf{GP}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(4) $\mathsf{FG}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(5) $\mathsf{NW}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(6) *For all n and k, $\Omega_{\mathbf{o}} \to (\Omega_{\mathbf{o}})^n_k$.*

For a topological space X let $\mathcal{D}$ denote the collection whose members are of the form $\mathcal{U}$, a family of open subsets of X, such that no element of $\mathcal{U}$ is dense in X, but $\bigcup\mathcal{U}$ is dense in X. And let $\mathcal{D}_\Omega$ be the set of $\mathcal{U} \in \mathcal{D}$ such that for each finite family $\mathcal{F}$ of nonempty open subsets of X there is a $U \in \mathcal{U}$ with $U \cap F \neq \emptyset$ for each $F \in \mathcal{F}$. The families $\mathcal{D}$ and $\mathcal{D}_\Omega$ were considered in [14] where it was proved that for X a set of real numbers, and $\mathsf{PR}(X)$ the Pixley–Roy space over X, the following holds:

Theorem 23. *If X is a set of real numbers, the following are equivalent for* $\mathsf{PR}(X)$:

(1) $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.
(2) *ONE has no winning strategy in the game* $\mathsf{G}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.
(3) *For each n and k $\mathcal{D}_\Omega \to (\mathcal{D}_\Omega)^n_k$.*

Each of these statements is equivalent to X having $\mathsf{S}_1(\Omega_X, \Omega_X)$.

Using the techniques above one can prove:

Theorem 24. *For a set X of reals the following are equivalent for* $\mathsf{PR}(X)$:

(1) $\mathsf{S}_1(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.
(2) $\mathsf{E}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.
(3) $\mathsf{GP}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.
(4) $\mathsf{FG}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.
(5) $\mathsf{NW}(\mathcal{D}_\Omega, \mathcal{D}_\Omega)$.
(6) *For all n and k, $\mathcal{D}_\Omega \to (\mathcal{D}_\Omega)^n_k$.*

For a noncompact topological space X call an open cover $\mathcal{U}$ a k-cover if there is for each compact $C \subset X$ a $U \in \mathcal{U}$ such that $C \subseteq U$, and if $X \notin \mathcal{U}$. Let $\mathcal{K}$ denote the collection of k-covers of such an X. If X is a separable metric space then each member of $\mathcal{K}$ has a countable subset which still is a member of $\mathcal{K}$. Using the techniques above one can prove:

Theorem 25. *For separable metric spaces X the following are equivalent*:

(1) *ONE has no winning strategy in* $\mathsf{G}_1(\mathcal{K}, \mathcal{K})$.
(2) $\mathsf{S}_1(\mathcal{K}, \mathcal{K})$.
(3) $\mathsf{E}(\mathcal{K}, \mathcal{K})$.
(4) $\mathsf{GP}(\mathcal{K}, \mathcal{K})$.
(5) $\mathsf{FG}(\mathcal{K}, \mathcal{K})$.

(6) $\mathsf{NW}(\mathcal{K}, \mathcal{K})$.
(7) *For all n and k*, $\mathcal{K} \to (\mathcal{K})^n_k$.

The equivalence of (2) and (7) for $n = 2$ and $k = 2$ is Theorem 8 of [2]. The equivalence of (1) and (2) is a result of [11]. The remaining equivalences are then derived as was done above for Ω.

A collection $\mathcal{C}$ of subsets of a set S is said to be a *combinatorial* ω*-cover* of S if $S \notin \mathcal{C}$, but for each finite subset F of S there is a $C \in \mathcal{C}$ with $F \subseteq C$. For an infinite cardinal number κ let Ω_κ be the set of *countable* combinatorial ω-covers of κ. Let $\mathsf{cov}(\mathcal{M})$ be the least infinite cardinal number κ such that the real line is a union of κ first category sets. By the Baire Category Theorem $\mathsf{cov}(\mathcal{M})$ is uncountable. Using the techniques of this paper one can prove:

Theorem 26. *For an infinite cardinal number κ the following are equivalent*:

(1) $\kappa < \mathsf{cov}(\mathcal{M})$.
(2) $\mathsf{S}_1(\Omega_\kappa, \Omega_\kappa)$.
(3) $\mathsf{E}(\Omega_\kappa, \Omega_\kappa)$.
(4) $\mathsf{GP}(\Omega_\kappa, \Omega_\kappa)$.
(5) $\mathsf{FG}(\Omega_\kappa, \Omega_\kappa)$.
(6) $\mathsf{NW}(\Omega_\kappa, \Omega_\kappa)$.
(7) *For all positive integers n and k*, $\Omega_\kappa \to (\Omega_\kappa)^n_k$.

4. Rothberger's property and forcing

Now we explore the connections between forcing and Rothberger's property. Much of this part of the paper is inspired by Theorem 9.3 of [1].

We begin by defining the following version of the well-known *Mathias reals* partially ordered set. Fix as before a countable ω-cover A of X, and enumerate it bijectively as $(a_n\colon n \in \mathbb{N})$. For $s \subset A$ finite, and $C \subset A|s$ with $C \in \Omega_X$, define:

$$\mathcal{M}_A := \{(s, C)\colon\ s \in [A]^{<\aleph_0} \text{ and } C \subset A|s \text{ and } C \in \Omega_X\}.$$

For (s_1, C_1) and (s_2, C_2) elements of $\mathcal{M}_A$, we define $(s_1, C_1) \prec (s_2, C_2)$ if: $s_2 \subset s_1$ and $C_1 \subset C_2$ and $s_1 \setminus s_2 \subset C_2|s_2$.

Now $(\mathcal{M}_A, \prec)$ is a partially ordered set. Its combinatorial and forcing properties are related to the combinatorial properties of ω-covers of X. In this section we will show (see Theorem 9.3 of [1]):

Theorem 27. *The following are equivalent:*

(1) $\mathsf{S}_1(\Omega_X, \Omega_X)$ *holds.*
(2) *For each countable $A \in \Omega_X$, for each sentence ψ in the $\mathcal{M}_A$-forcing language, and for each $(s, B) \in \mathcal{M}_A$, there is a $C \subset B$ with $C \in \Omega_X$ such that $(s, C) \Vdash \psi$, or $(s, C) \Vdash \neg\psi$.*

4.1. *Proof of* (1) $\Rightarrow$ (2)

Fix a sentence ψ of the $\mathcal{M}_A$-forcing language and fix $(s, B) \in \mathcal{M}_A$. Define the subsets

$$\mathcal{W} = \bigcup\{[t, C]\colon\ (t, C) \in \mathcal{M}_A \text{ and } (t, C) \Vdash \psi\}$$

and

$$\mathcal{D} = \bigcup\{[t, C]\colon\ (t, C) \in \mathcal{M}_A \text{ and } (t, C) \Vdash \neg\psi\}.$$

Then $\mathcal{W}$ and $\mathcal{D}$ are open sets in the Ellentuck topology on $[A]^{\aleph_0} \cap \Omega_X$. Moreover, by Corollary VII.3.7(a) of [8], $\mathcal{R} = \mathcal{W} \cup \mathcal{D}$ is dense. By Theorem 17, $\mathcal{R}$, $\mathcal{W}$ and $\mathcal{D}$ are completely Ramsey. Thus, for the given $(s, B) \in \mathcal{M}_A$ there is a $B_1 \in [B]^{\aleph_0} \cap \Omega_X$ such that $[s, B_1] \subset \mathcal{R}$, or $[s, B_1] \cap \mathcal{R} = \emptyset$; since $\mathcal{R}$ is dense and $[s, B_1]$ is nonempty and open we have $[s, B_1] \subset \mathcal{R}$. But now $\mathcal{W}$ is completely Ramsey and so there is a $B_2 \in [B_1]^{\aleph_0} \cap \Omega_X$ with $[s, B_2] \subset \mathcal{W}$, or $[s, B_2] \cap \mathcal{W} = \emptyset$. Since $[s, B_2] \subset [s, B_1]$, we have that $[s, B_2] \subset \mathcal{W}$ or $[s, B_2] \subset \mathcal{D}$. In either case we have $(s, B_2) \Vdash \psi$, or $(s, B_2) \Vdash \neg\psi$.

4.2. *Proof of* (2) $\Rightarrow$ (1)

This proof takes more work. We show that in fact (2) implies that $\Omega_X \to (\Omega_X)^2_2$ holds. To see this, assume on the contrary that $\Omega_X \to (\Omega_X)^2_2$ fails. Choose a countable $A \in \Omega_X$ and a function $f : [A]^2 \to \{0, 1\}$ which witness this failure. Enumerate A bijectively as $(a_n\colon n \in \mathbb{N})$ and build the following corresponding partition tree.

$T_\emptyset = A$. $T_{(i)} := \{a_n\colon\ n > 1 \text{ and } f(\{a_1, a_n\}) = i\}$. For $\sigma \in^{<\omega} \{0, 1\}$ of length m for which $T_\sigma \in \Omega_X$, $T_{\sigma\frown(i)} := \{a_n \in T_\sigma\colon n > m \text{ and } f(\{a_m, a_n\}) = i\}$.

Observe that for each σ with $T_\sigma \in \Omega_X$ we have $T_{\sigma\frown(0)} \in \Omega_X$ or $T_{\sigma\frown(1)} \in \Omega_X$. For each n, define $\mathcal{T}_n := \{T_\sigma \in \Omega_X\colon length(\sigma) = n\}$. Then we have from the definitions that:

(1) For each $B \in [A]^{\aleph_0} \cap \Omega_X$ and for each n there is a $T \in \mathcal{T}_n$ with $B \cap T \in \Omega_X$.
(2) For each n, for each $T \in \mathcal{T}_{n+1}$ there is a unique $T' \in \mathcal{T}_n$ with $T \subset T'$.
(3) For each n and σ, if $a_n \in T_\sigma$, then $n > m = length(\sigma)$.

Claim 1. If there is a $B \in [A]^{\aleph_0} \cap \Omega_X$ such that for each n there is a $T \in \mathcal{T}_n$ with $B \setminus \{a_j\colon j \leqslant n\} \subset T$, then there is a $C \in [A]^{\aleph_0} \cap \Omega_X$ such that f is constant on $[C]^2$.

For let such a B be given. Since the elements of $\mathcal{T}_1$ are pairwise disjoint, choose the unique $i_1 \in \{0,1\}$ with $B \setminus \{a_1\} \subset T_{(i_1)}$. Letting T be the unique element of $\mathcal{T}_2$ with $B \setminus \{a_1, a_2\} \subset T$, we see that $T \subset T_{(i_1)}$, and so for a unique $i_2 \in \{0,1\}$, $B \setminus \{a_1, a_2\} \subset T_{(i_1,i_2)}$. Arguing like this we find an infinite sequence $(i_j : j < \infty)$ in ${}^{\mathbb{N}}\{0,1\}$ such that for each m, $B \setminus \{a_1, \dots, a_m\} \subset T_{(i_1,\dots,i_m)}$.

Write $B = \{a_{n_j}\colon j \in \mathbb{N}\}$ where $n_i < n_j$ whenever $i < j$. Put $B_1 = \{a_{n_j}\colon i_{n_j} = 1\}$ and $B_0 = \{a_{n_j}\colon i_{n_j} = 0\}$. Then $B_0 \in \Omega_X$, or $B_1 \in \Omega_X$. In the former case f is constant of value 0 on $[B_0]^2$, and in the latter case f is constant of value 1 on $[B_1]^2$. This completes the proof of Claim 1.

Note that the conclusion of Claim 1 holds also if instead we hypothesize that $B \in [A]^{\aleph_0} \cap \Omega_X$ is such that for each n with $a_n \in B$ there is a $T \in \mathcal{T}_n$ with $B \setminus \{a_j\colon j \leqslant n\} \subset T$.

Since we are assuming that there is no $B \in [A]^{\aleph_0} \cap \Omega_X$ with f constant on $[B]^2$, we get: There is no $B \in [A]^{\aleph_0} \cap \Omega_X$ such that for each n with $a_n \in B$ there is a $T \in \mathcal{T}_n$ with $B \setminus \{a_j\colon j \leqslant n\} \subset T$. Indeed, this is equivalent to:

For each $B \in [A]^{\aleph_0} \cap \Omega_X$ there is an n with $a_n \in B$ but for each $T \in \mathcal{T}_n$ we have $B \setminus \{a_j\colon j \leqslant n\} \not\subset T$.

In what follows we will use $\dot{a}$ to denote the canonical name of the ground model object a in the forcing language. Define the $\mathcal{M}_A$-name

$$\Gamma := \big\{\big(\dot{a}_n, (s, B)\big)\colon (s, B) \in \mathcal{M}_A \text{ and } a_n \in s\big\}.$$

Then for each $\mathcal{M}_A$-generic filter G we have

$$\Gamma_G = \bigcup \big\{s \in [A]^{<\aleph_0}\colon \big(\exists B \in [A]^{\aleph_0} \cap \Omega_X\big)\big((s, B) \in G\big)\big\}.$$

Claim 2. $(\emptyset, A) \Vdash$ "$(\exists n)(\forall T \in \dot{\mathcal{T}}_n)(\Gamma \setminus \{\dot{a}_j\colon j \leqslant n\} \not\subseteq T)$".

For suppose that on the contrary $(s, B) \Vdash$ "$(\forall n)$ $(\exists T \in \dot{\mathcal{T}}_n)$ $(\Gamma \setminus \{\dot{a}_j\colon j \leqslant n\} \subseteq T)$". Since we have $B \in [A]^{\aleph_0} \cap \Omega_X$, choose an n_1 so that $B \setminus \{a_j\colon j \leqslant n_1\}$ is not a subset of any $T \in \mathcal{T}_n$. Then choose a $T_{n_1} \in \mathcal{T}_{n_1}$ so that $B \cap T_{n_1} \in \Omega_X$. Also choose $a_m \in B \setminus (T_{n_1} \cup \{a_j\colon j \leqslant n_1\})$. Put $B' = B | \{a_j\colon j \leqslant m_1\}$ and put $t = s \cup \{a_m\}$. Then as $(s, B) \Vdash$ "$\dot{\mathcal{T}}_{n_1}$ is a disjoint family" and $(t, B' \cap T_{n_1}) \Vdash$ "$(\Gamma \setminus \{\dot{a}_j\colon j \leqslant \dot{n}_1\}) \cap \dot{T}_{n_1} \neq \emptyset$".

$$(t, B' \cap T_{n_1}) \Vdash \text{“}\Gamma \setminus \big\{\dot{a}_j\colon j \leqslant \dot{n}_1\big\} \subset \dot{T}_{n_1}\text{”}. \tag{1}$$

But evidently we also have

$$(t, B' \cap T_{n_1}) \Vdash \text{“}\dot{a}_m \in \big(\Gamma \setminus \big\{\dot{a}_j\colon j \leqslant \dot{n}_1\big\}\big) \setminus \dot{T}_{n_1}\text{”}. \tag{2}$$

Thus we have a condition forcing contradictory statements, a contradiction. It follows that Claim 2 holds.

Now we construct a sentence $\Psi(\Gamma)$ in the forcing language:

$$\text{“}\Gamma \cap \big\{\dot{a}_j\colon j < n\big\} \text{ is even for the least } n \text{ with } \dot{a}_n \in \Gamma \text{ and for all } T \in \dot{\mathcal{T}}_n\ \Gamma \setminus \big\{\dot{a}_j\colon j \leqslant n\big\} \not\subset T\text{”}.$$

By hypothesis (2) of the theorem, choose a $B \in [A]^{\aleph_0} \cap \Omega_X$ such that $(\emptyset, B)$ decides $\Psi(\Gamma)$.

Choose k_1 minimal so that $a_{k_1} \in B$ and for each $T \in \mathcal{T}_{k_1}$ we have $B \setminus \{a_j\colon j \leqslant k_1\} \not\subseteq T$. Put $B_1 = B \setminus \{a_j\colon j \leqslant k_1\}$ and choose $T_{k_1} \in \mathcal{T}_{k_1}$ so that $C_1 := B_1 \cap T_{k_1} \in \Omega_X$. Choose ℓ_1 so that $a_{\ell_1} \in B_1 \setminus T_{k_1}$. Then $(\{a_{k_1}, a_{\ell_1}\}, C_1) < (\emptyset, B)$ and so also $(\{a_{k_1}, a_{\ell_1}\}, C_1)$ decides $\Psi(\Gamma)$.

By the construction of C_1 we see that for $T' \in \mathcal{T}_{k_1} \setminus \{T_{k_1}\}$, also $(\{a_{k_1}, a_{\ell_1}\}, C_1) \Vdash$ "$\Gamma \setminus \{\dot{a}_j\colon j \leqslant \dot{k}_1\} \not\subseteq \dot{T}'$". And since $a_{\ell_1} \notin T_{k_1}$ we also have $(\{a_{k_1}, a_{\ell_1}\}, C_1) \Vdash$ "$\Gamma \setminus \{\dot{a}_j\colon j \leqslant \dot{k}_1\} \not\subseteq \dot{T}_{k_1}$". Moreover, $(\{a_{k_1}, a_{\ell_1}\}, C_1) \Vdash$ "$\Gamma \cap \{\dot{a}_j\colon j < \dot{k}_1\} = \emptyset$". Since k_1 was chosen minimal and $a_{k_1} \in B$, the least n having the properties of k_1 is k_1. It follows that $(\{a_{k_1}, a_{\ell_1}\}, C_1) \Vdash \Psi(\Gamma)$, and as

$(\emptyset, B)$ already decides $\Psi(\Gamma)$, we have

$$(\emptyset, B) \Vdash \Psi(\Gamma). \tag{3}$$

Now repeat the previous construction starting with C_1 in place of B. Choose k_2 minimal so that $a_{k_2} \in C_1$ and for each $T \in \mathcal{T}_{k_2}$ we have $C_1 \setminus \{a_j\colon\ j \leqslant k_2\} \nsubseteq T$. Since $a_{k_1} \notin C_1$, we have $k_2 > k_1$. Put $B_2 = C_1 \setminus \{a_j\colon\ j \leqslant k_2\}$ and choose $T_{k_2} \in \mathcal{T}_{k_2}$ so that $C_2 := B_2 \cap T_{k_2} \in \Omega_X$. Choose ℓ_2 so that $a_{\ell_2} \in B_2 \setminus T_{k_2}$. Then $(\{a_{k_1}, a_{k_2}, a_{\ell_2}\}, C_2) < (\emptyset, B)$ and so also $(\{a_{k_1}, a_{k_2}, a_{\ell_2}\}, C_2)$ decides $\Psi(\Gamma)$. By the construction of C_2 we see that for $T' \in \mathcal{T}_{k_2} \setminus \{T_{k_2}\}$, also $(\{a_{k_1}, a_{k_2}, a_{\ell_2}\}, C_2) \Vdash$ "$\Gamma \setminus \{\dot{a}_j\colon\ j \leqslant \dot{k}2\} \nsubseteq \dot{T}'$". And since $a_{\ell_2} \notin T_{k_2}$ we also have $(\{a_{k_1}, a_{k_2}, a_{\ell_2}\}, C_2) \Vdash$ "$\Gamma \setminus \{\dot{a}_j\colon\ j \leqslant \dot{k}2\} \nsubseteq \dot{T}_{k_2}$". By minimality of k_2 and the fact that $a_{k_2} \in C_2$, we get that the minimal n with these properties of k_2 is k_2. However, $(\{a_{k_1}, a_{k_2}, a_{\ell_2}\}, C_2) \Vdash$ "$\Gamma \cap \{\dot{a}_j\colon\ j \leqslant \dot{k}2\} = \{\dot{a}_{k_1}\}$". This means that $(\{a_{k_1}, a_{k_2}, a_{\ell_2}\}, C_2) \Vdash \neg\Psi(\Gamma)$. Since $(\{a_{k_1}, a_{k_2}, a_{\ell_2}\}, C_2) < (\emptyset, B)$ and $(\emptyset, B)$ already decides $\Psi(\Gamma)$, we find that

$$(\emptyset, B) \Vdash \neg\Psi(\Gamma). \tag{4}$$

Since (3) and (4) yield a contradiction, the hypothesis that $\Omega \to (\Omega)^2_2$ fails is false. This completes the proof of (2) $\Rightarrow$ (1) of Theorem 27.

Remark. The above result is again given for Ω, but a study of the proofs will reveal that these equivalences hold for several other families $\mathcal{A}$, including the examples mentioned earlier. Theorem 27 has several consequences that will be explored elsewhere. One of the mentionable consequences is that forcing with $\mathcal{M}_A$ preserves cardinals, and in the generic extension the only groundmodel sets of reals having $\mathsf{S}_1(\Omega, \Omega)$ are the countable sets. And a countable support iteration of length $\aleph_2$ over a ground model satisfying the Continuum Hypothesis gives a model of Borel's Conjecture, just like the usual Mathias reals iteration does—[1].

In closing: Analogous results can be proved for the selection principle $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{A})$ and its relatives. These will be reported elsewhere.

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Appendix A. Regarding Theorem 21

Strictly speaking, the only equivalence that has been explicitly proved in the literature is the equivalence of $\mathsf{S}_1(\Omega, \Omega)$ with $\Omega \to (\Omega)^2_2$, with the remark that the techniques used to prove this case yield by an induction the full version that for all finite n and k we have $\Omega \to (\Omega)^n_k$. It is perhaps worth putting down the main elements of such an argument explicitly for future reference. The only implication we need to prove is the implication that $\Omega \to (\Omega)^2_2$ implies that for all n and k $\Omega \to (\Omega)^n_k$.

Claim 1. $\Omega \to (\Omega)^2_2$ implies that for each $k > 1$, $\Omega \to (\Omega)^2_k$.

This can be done by induction on $k+1$. For $k = 1$ this is the hypothesis. Assuming we have proven the implication for $j \leqslant k$, consider a countable ω-cover $\mathcal{U}$ of X and a coloring $f : [\mathcal{U}]^2 \to \{1, \ldots, k+1, k+2\}$. Define a new coloring g so that

$$g\bigl(\{U, V\}\bigr) = \begin{cases} f(\{U, V\}) & \text{if } f(\{U, V\}) < k+1, \\ k+1 & \text{otherwise.} \end{cases}$$

Applying the induction hypothesis we find an ω-cover $\mathcal{V} \subset \mathcal{U}$ and an $i \in \{1, \ldots, k+1\}$ such that $g(\{U, V\}) = i$ for all $\{U, V\} \in \mathcal{V}^2$. If $i < k+1$ then indeed $\mathcal{V}$ works for f. Else, $\mathcal{V}$ is an ω-cover on whose pairs f takes values $k+1$ or $k+2$, and now apply $\Omega \to (\Omega)^2_2$.

Claim 2. For $n > 2$ and $k > 1$, $\Omega \to (\Omega)^n_k$ implies $\Omega \to (\Omega)^2_k$.

This can be done by starting with a countable ω-cover $\mathcal{U}$ and a coloring $f : [\mathcal{U}]^2 \to \{1, \ldots, k\}$. Enumerate $\mathcal{U}$ bijectively as $\{U_m\colon\ m \in \mathbb{N}\}$. Define $g : [\mathcal{U}]^n \to \{1, \ldots, k\}$ by

$$g\bigl(\{U_{i_1}, \ldots, U_{i_n}\}\bigr) = f\bigl(\{U_{i_1}, U_{i_2}\}\bigr),$$

where we list the n-tuples according to increasing index in the chosen enumeration. Apply $\Omega \to (\Omega)^n_k$.

Claim 3. For $n > 1$ and $k > 1$, $\Omega \to (\Omega)^n_k$ implies $\Omega \to (\Omega)^{n+1}_k$.

To prove this we use the fact that For $n > 1$ and $k > 1$, $\Omega \to (\Omega)^n_k$ implies $\Omega \to (\Omega)^2_2$, which in turn implies that ONE has no winning strategy in the game $\mathsf{G}_1(\Omega, \Omega)$.

Let a countable ω-cover $\mathcal{U}$ be given, as well as $f : [\mathcal{U}]^{n+1} \to \{1, \dots, k\}$. Enumerate $\mathcal{U}$ bijectively as $\{U_m\colon m \in \mathbb{N}\}$. Define a strategy F for ONE in the game $\mathsf{G}_1(\Omega, \Omega)$ as follows:

Fix U_1 and define

$$g_1 : \left[\mathcal{U} \setminus \{U_1\}\right]^n \to \{1, \dots, k\}$$

by $g_1(\mathcal{V}) = f(\{U_1\} \cup \mathcal{V})$. Using $\Omega \to (\Omega)^n_k$, fix an $i_1 \in \{1, \dots, k\}$ and an ω-cover $\mathcal{U}_1 \subset \mathcal{U}$ such that $g_1(\mathcal{V}) = i_1$ for each $\mathcal{V} \in [\mathcal{U}_1]^n$. Declare ONE's move to be $F(\emptyset) = \mathcal{U}_1$.

When TWO responds with $T_1 = U_{n_1} \in F(\emptyset)$, ONE first defines

$$g_2 : \left[\mathcal{U}_1 \setminus \{U_j\colon\ j \leqslant n_1\}\right]^n \to \{1, \dots, k\}$$

by $g_2(\mathcal{V}) = f(\{U_{n_1}\} \cup \mathcal{V})$. Then, using $\Omega \to (\Omega)^n_k$, fix an $i_{n_1} \in \{1, \dots, k\}$ and an ω-cover $\mathcal{U}_2 \subset \mathcal{U}_1 \setminus \{U_j\colon\ j \leqslant n_1\}$ such that $g_2(\mathcal{V}) = i_{n_1}$ for each $\mathcal{V} \in [\mathcal{U}_2]^n$. Declare ONE's move to be $F(T_1) = \mathcal{U}_2$.

When TWO responds with $T_2 = U_{n_2} \in F(T_1)$, ONE first defines

$$g_3 : \left[\mathcal{U}_2 \setminus \{U_j\colon\ j \leqslant n_2\}\right]^n \to \{1, \dots, k\}$$

by $g_3(\mathcal{V}) = f(\{U_{n_2}\} \cup \mathcal{V})$. Then, using $\Omega \to (\Omega)^n_k$, fix an $i_{n_2} \in \{1, \dots, k\}$ and an ω-cover $\mathcal{U}_3 \subset \mathcal{U}_2 \setminus \{U_j\colon\ j \leqslant n_2\}$ such that $g_3(\mathcal{V}) = i_{n_2}$ for each $\mathcal{V} \in [\mathcal{U}_3]^n$. Declare ONE's move to be $F(T_1, T_2) = \mathcal{U}_3$.

This describes ONE's strategy in this game. Since it is not winning for ONE, we find a play $F(\emptyset), T_1, F(T_1), T_2, F(T_1, T_2), T_3, \dots$ which is lost by ONE. Associated with this play we have an increasing infinite sequence $n_1 < n_2 < \cdots < n_k < \cdots$ for which $T_k = U_{n_k}$, all k, and a sequence $i_{n_k}, k \in \mathbb{N}$ of elements of $\{1, \dots, k\}$, and a sequence $\mathcal{U}_n, n \in \mathbb{N}$, of ω-covers such that:

(1) For each m, $T_m = U_{n_m} \in \mathcal{U}_m \subset \mathcal{U}_{m-1} \setminus \{U_{n_j}\colon\ j \leqslant m-1\}$.
(2) For each m, $f(\{T_m\} \cup \mathcal{V}) = i_{n_m}$ whenever $\mathcal{V} \in [\mathcal{U}_{m+1}]^n$.
(3) $\{T_m\colon\ m \in \mathbb{N}\} \subset \mathcal{U}$ is an ω-cover.

Fix an i such that $\mathcal{W} = \{T_m\colon\ i_{n_m} = i \text{ and } m > n\}$ is an ω-cover. Then for each $\mathcal{V} \in [\mathcal{W}]^{n+1}$ we have $f(\mathcal{V}) = i$.

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